Exercise 26

Prove the statement using the precise definition of a limit.

$$\lim_{x \to 0} \sqrt[3]{x} = 0$$

Solution

Proving this limit is logically equivalent to proving that

$$\text{if} \quad |x - 0| < \delta \qquad \text{then} \qquad |\sqrt[3]{x} - 0| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x|$.

$$|\sqrt[3]{x} - 0| < \varepsilon$$

$$|\sqrt[3]{x}| < \varepsilon$$

$$\sqrt[3]{|x|} < \varepsilon$$

$$\left(\sqrt[3]{|x|}\right)^3 < (\varepsilon)^3$$

$$|x| < \varepsilon^3$$

Choose $\delta = \varepsilon^3$. Now, assuming that $|x| < \delta$,

$$\begin{aligned} |\sqrt[3]{x} - 0| &= |\sqrt[3]{x}| \\ &= \sqrt[3]{|x|} \\ &< \sqrt[3]{\delta} \\ &= \sqrt[3]{\varepsilon^3} \\ &= \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x \to 0} \sqrt[3]{x} = 0.$$